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Flatness of Tangent Cones of a Family of Hypersurfaces

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The notion of the "tangent cone" at a singular point of an algebraic variety ought to generalize that of the tangent space at a nonsingular point. In [8] Whitney proposed various candidates, including the usual one. The usual tangent cone stems from the geometric notion of a secant line; as a set the tangent cone at a point p consists of the limiting directions of lines determined by p and by a mobile point q, as q approaches p. One obtains a different notion by allowing two points q_1 and q_2 to be mobile; following Johnson [3] we call the resulting set of limiting lines (as q_1 and q_2 simultaneously approach p) the "tangent star" of the variety at the point. An even larger "tangent cone" is the Zariski tangent space.

For a nonsingular variety X the tangent space at a point is the fiber of the tangent bundle TX. It is natural to ask whether the tangent cone at a point of a singular variety (or other algebraic scheme) is, in a similar fashion, the fiber of some variety (or scheme) over the base. For the usual tangent cone, the answer is unsatisfactory: for example, at the vertex of the cone on a plane curve the projectivized tangent cone is a copy of the curve, whereas by continuity it ought to be a projective plane. For the tangent star and the Zariski tangent space, however, there are satisfactory global schemes, which we call the "tangent star cone" and the "Zariski cone." The definitions are deceptively simple (see Section 1 below); determining explicit equations for a tangent star cone can be an intricate matter. In this paper we find these equations for hypersurfaces, in particular for non-reduced hypersurfaces.

The definitions of the tangent star cone and the Zariski cone generalize without difficulty to a relative situation, namely that of a scheme $\mathscr{X}$ over a base scheme T. In fact to analyze the static situation, i.e., to determine these cones for a single hypersurface, we find it necessary to work with families of hypersurfaces. What we discover in the relative case is a

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remarkable flatness: if the components of the hypersurface $\mathscr{X}$ "do not coalesce" over a point of T then the relative tangent star cone and the relative Zariski cone are flat over T above that point. In particular these cones are flat if the fiber over the point is a reduced hypersurface; the presence of singularities is irrelevant.

The situation for families of varieties of higher codimension, even complete intersections, appears more complicated.

Our results generalize those of Varley and Yokura, who in [7] analyze reduced hypersurfaces. They, Clint McCrory, Ragni Piene, Ted Shifrin, and the referee have made many helpful suggestions. Similar situations have been studied by Herzog, Simis, and Vasconcelos [2]. Theorem 3 was conjectured after many calculations using Bayer and Stillman's Macaulay [6].

In this paper we assume that all schemes are defined over a field k of characteristic zero, and that they are separated and of finite type.

1. Tangent Star Cones and Zariski Cones

Consider a scheme X embedded as the diagonal in $X \times X$. The *tangent star cone* $TS(X)$ is the normal cone. If the diagonal is defined by the ideal sheaf $\mathscr{I}$, then

$$TS(X) = \operatorname{Spec} \bigoplus_{j \geq 0} \frac{\mathscr{I}^j}{\mathscr{I}^{j+1}}.$$

The *projectivized tangent star cone* is

$$\mathbb{P}TS(X) = \operatorname{Proj} \bigoplus_{j \geq 0} \frac{\mathscr{I}^j}{\mathscr{I}^{j+1}}.$$

It is the exceptional divisor of the blowup of $X \times X$ along its diagonal.

If X is a subscheme of $\mathbb{A}^n$ then $\mathbb{P}TS(X)$ may be obtained by the following "limiting secants" construction. Letting $\overline{p_1 p_2}$ denote the line through the origin parallel to the line through two distinct points p_1 and p_2, we embed $(X \times X) - X$ in $\mathbb{A}^n \times \mathbb{A}^n \times \mathbb{P}^{n-1}$ by

$$(p_1, p_2) \mapsto (p_1, p_2, \overline{p_1 p_2}),$$

and form the closure of the image. The resulting scheme over the diagonal is $\mathbb{P}TS(X)$. The fiber over a point consists (as a set) of limiting secants; Johnson [3] dubbed it the *tangent star*.

Example. Let X be the union of the three coordinate axes in $\mathbb{A}^3$, denoted X_1, X_2, X_3. Since X is nowhere dense in $X \times X$, the blowup $Bl_X(X \times X)$ has nine irreducible components. Its fundamental class is

$$[Bl_X(X \times X)] = \sum_{k,l=1}^{3} [Bl_{X_k \cap X_l}(X_k \times X_l)].$$

Hence the fundamental class of the projectivized tangent star cone is

$$[\mathbb{P}TS(X)] = \sum_{k,l=1}^{3} [\mathbb{P}C_{X_k \cap X_l}(X_k \times X_l)],$$

where the component varieties on the right are projectivized normal cones. Interchanging k and l does not change the component, so the projectivized tangent star has six components. Three of these components (those for which $k \neq l$) map set-theoretically to the origin. Each of the three is a $\mathbb{P}^1$ consisting of the limiting secants coming from a pair of axes, and each occurs in $\mathbb{P}TS(X)$ with multiplicity 2. (This analysis ignores the possibility of embedded components.)

EXAMPLE. Suppose that X is the r-fold origin $x^r = 0$ in $\mathbb{A}^1$. Theorem 3 of the next section will show that $TS(X)$ is defined in $\mathbb{A}^2$ by $x^r = 0$ and by

$$x^{r-m}u^{2m-1} = 0, \qquad 1 \leqslant m \leqslant r.$$

Note that this is a scheme supported at the origin.

A component of the tangent star cone may be contained in its zero section, as the previous example shows. We call such a component *irrelevant*, since its defining ideal contains the irrelevant ideal sheaf. Irrelevant component are discarded when we pass to the projectivized tangent star cone. They can be recovered from a blowup by the device of replacing X by $X \times \mathbb{A}^1$. Indeed, $TS(X \times \mathbb{A}^1)$ is isomorphic to $TS(X) \times \mathbb{A}^2$, and has no irrelevant components; it is locally defined by the same equations as its projectivization $\mathbb{P}TS(X \times \mathbb{A}^1)$, which is the exceptional divisor of the blowup of $(X \times \mathbb{A}^1) \times (X \times \mathbb{A}^1)$ along its diagonal. (In the example, we now regard $x^r = 0$ as describing a thickened line in $\mathbb{A}^2$. The projectivized tangent star is defined in $\mathbb{A}^2 \times \mathbb{P}^1$ by the same equations as above, with two variables tacit.)

The tangent star cone is a closed subscheme of the *Zariski cone* $Z(X)$, the cone associated to the sheaf of Kähler differentials:

$$Z(X) = \operatorname{Spec} \operatorname{Sym} \frac{\mathscr{I}}{\mathscr{I}^2}.$$

The fiber of this cone over a point of X is the Zariski tangent space of X at that point.

If $\mathscr{X}$ is a scheme over a nonsingular variety T, the *relative tangent star cone* $TS(\mathscr{X}/T)$ is the normal cone of the relative diagonal in $\mathscr{X} \times_T \mathscr{X}$. If $\mathscr{I}$ now denotes the ideal sheaf of the relative diagonal, then

$$TS(\mathscr{X}/T) = \operatorname{Spec} \bigoplus_{j \geqslant 0} \frac{\mathscr{I}^j}{\mathscr{I}^{j+1}}.$$

The *relative Zariski cone* $Z(\mathscr{X}/T)$ is the cone associated to the sheaf of relative Kähler differentials:

$$Z(\mathscr{X}/T) = \operatorname{Spec} \operatorname{Sym} \frac{\mathscr{I}}{\mathscr{I}^2}.$$

As in the static case $T = \operatorname{Spec} k$, one can recover the irrelevant components of the tangent star cone from a blowup by the device of replacing $\mathscr{X}$ by $\mathscr{X} \times \mathbb{A}^1$.

If $\mathscr{X}$ is smooth over T then the conormal sheaf $\mathscr{I}/\mathscr{I}^2$ is locally free and isomorphic to the relative tangent sheaf of $\mathscr{X}$ over T. Hence in this case $Z(\mathscr{X}/T)$ is isomorphic to the total space $T(\mathscr{X}/T)$ of the relative tangent bundle. Furthermore

$$\bigoplus_{j \geqslant 0} \frac{\mathscr{I}^j}{\mathscr{I}^{j+1}} \cong \operatorname{Sym} \frac{\mathscr{I}}{\mathscr{I}^2};$$

hence $TS(\mathscr{X}/T) = Z(\mathscr{X}/T)$.

If $\mathscr{X}$ is a closed subscheme of another scheme $\mathscr{Y}$ over T, then the Zariski and tangent star cones of $\mathscr{X}$ are contained in the respective cones of $\mathscr{Y}$. In particular if $\mathscr{Y}$ is smooth over T, then $Z(\mathscr{X}/T)$ and $TS(\mathscr{X}/T)$ are closed subschemes of $T(\mathscr{Y}/T)|_{\mathscr{X}}$, the total space of the restriction to $\mathscr{X}$ of the relative tangent bundle.

2. The Hypersurface Case

Suppose that $\mathscr{Y}$ is smooth over the nonsingular variety T. Suppose that $\mathscr{X}$ is a hypersurface in $\mathscr{Y}$ flat over T, i.e., that $\mathscr{X}$ is a flat family of hypersurfaces. On each chart of $\mathscr{Y}$, $\mathscr{X}$ is defined by the vanishing of a regular function

$$f = \prod_{k=1}^{s} f_k^{r_k},$$

where the f_k's are reduced and relatively prime. (We could assume that the f_k's are irreducible, but choose not to.)

Let t denote a local system of coordinates at a point O of T. If the specializations of the f_k's obtained by setting $t=0$ are likewise reduced and relatively prime, we say that *the components of* $\mathscr{X}$ *do not coalesce over* O *in this chart.* Note that this definition does not depend on the choice of f_k's. If there is a collection of such charts covering the fiber of $\mathscr{X}$ over O then we say that *the components of* $\mathscr{X}$ *do not coalesce over* O.

EXAMPLES. The components of $xy=t$ (one-parameter family of plane curves) do not coalesce over $t=0$. The components of $y^2=tx^2$ do coalesce over $t=0$.

EXAMPLE. The components of $\mathscr{X}$ do not coalesce if X_O, the fiber of $\mathscr{X}$ over O, is reduced; here we have a flat family of reduced hypersurfaces.

THEOREM 1. *Suppose that* $\mathscr{Y}$ *is smooth over the nonsingular variety* T, *and that* $\mathscr{X}$ *is a hypersurface in* $\mathscr{Y}$ *flat over* T. *Then the fiber over a point* $O\in T$ *of the relative Zariski cone* $Z(\mathscr{X}/T)$ *is* $Z(X_O)$.

Suppose furthermore that the components of $\mathscr{X}$ *do not coalesce over* O. *Then on the fiber over* O *the relative Zariski cone is flat over* T; *on the fiber over* O *the relative tangent star cone* $TS(\mathscr{X}/T)$ *is flat over* T; *and the fiber over* O *of* $TS(\mathscr{X}/T)$ *is* $TS(X_O)$.

THEOREM 2. *Conversely, if the components of* $\mathscr{X}$ *do coalesce over* O, *then on the fiber over* O *the relative tangent star cone is not flat, and its fiber is not* $TS(X_0)$.

The proof of Theorem 1 depends on a knowledge of explicit local equations for $TS(\mathscr{X}/T)$ and $Z(\mathscr{X}/T)$, which we now proceed to describe.

Suppose that $\mathscr{Y}$ is a closed subscheme of $\mathbb{A}^n\times T$. Let $x_1, ..., x_n$ be coordinates on the affine space, and t a coordinate system on T. Let $u_1, ..., u_n$ be coordinates for the tangent bundle with respect to the basis $\partial/\partial x_1, ..., \partial/\partial x_n$. Let P denote the partial differential operator

$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial x_i}.$$

Following [5], we call $P^d f$ the *dth polarization* of f; it is homogeneous of degree d in $u_1, ..., u_n$.

Suppose that $\mathscr{X}$ is defined in $\mathscr{Y}$ by

$$0=f(x_1, x_2, ..., x_n, t)=\prod_{k=1}^{s} f_k^{r_k},$$

with the f_k's reduced and relatively prime. For each positive integer m write f as a product

$$f = \prod_{r_k < m} f_k^{r_k} \cdot \prod_{r_k \geqslant m} f_k^{r_k}$$

in which the first product is indexed by all k for which the exponent r_k is less than m and the second by all k for which r_k is at least equal to m. We define

$$S_m f = \left(\prod_{r_k < m} f_k^{r_k} \right)^2 \cdot P^{2m-1} \left(\prod_{r_k \geqslant m} f_k^{r_k + m - 1} \right).$$

Note in particular that $S_1 f = Pf$ and that $S_m f = 0$ for m sufficiently large. Also note that in general $S_m f$ depends on the choice of f and the choice of factorization; for example, it can be altered simply by multiplying some f_k by a unit.

THEOREM 3. *Suppose that $\mathscr{Y} \subset \mathbb{A}^n \times T$ is smooth over the nonsingular variety T, and that $\mathscr{X}$ is a hypersurface in $\mathscr{Y}$, flat over T, defined in $\mathscr{Y}$ by*

$$0 = f = \prod_{k=1}^{s} f_k^{r_k},$$

with the f_k's reduced and relatively prime. Then the Zariski cone $Z(\mathscr{X}/T)$ is defined inside $T(\mathscr{Y}/T)|_{\mathscr{X}}$ by the vanishing of $S_1 f$, i.e., of Pf. The tangent star cone $TS(\mathscr{X}/T)$ is defined by the vanishing of all $S_m f$.

EXAMPLE. If $\mathscr{X}$ is reduced then $Z(\mathscr{X}/T) = TS(\mathscr{X}/T)$.

EXAMPLE. Let $\mathscr{X}$ be the 1-parameter family of affine plane curves defined by $xy = t$. Then $Z(\mathscr{X}/T) = TS(\mathscr{X}/T)$ is given in $\mathbb{A}^5$ by

$$xy = t, \qquad yu + xv = 0.$$

Its fiber over $t = 0$ is $Z(X_O) = TS(X_O)$:

$$xy = yu + xv = 0.$$

The primary decomposition

$$(xy,\ yu + xv) = (x, u) \cap (y, v) \cap (x^2, xy, y^2, yu + xv)$$

shows that $TS(X_O)$ has three components. The component over the origin occurs with multiplicity 2.

EXAMPLE. Define $\mathscr{X}$ by $y^2 = tx^2$; $Z(\mathscr{X}/T) = TS(\mathscr{X}/T)$ is given by

$$y^2 = tx^2, \qquad yv = txu.$$

Its fiber over $t = 0$ is

$$y^2 = yv = 0,$$

which is the same as $Z(X_O)$. The relative tangent star cone is certainly not flat, because the dimension of the fiber over $t = 0$ is too large. The tangent star cone of X_O is

$$y^2 = yv = v^3 = 0.$$

The flattening of $TS(\mathscr{X}/T)$, i.e., the unique subscheme flat over the parameter space and having the same generic fiber, is

$$y^2 = tx^2, \qquad yv = txu, \qquad xyu = x^2v,$$

with fiber over $t = 0$ given by

$$y^2 = yv = 0, \qquad xyu = x^2v.$$

This fiber and $TS(X_O)$ are incomparable: neither scheme contains the other.

EXAMPLE. For $xy^3 = 0$, $Z(X)$ is given by

$$xy^3 = y^3u + 3xy^2v = 0,$$

which is not even equidimensional. The tangent star cone $TS(X)$ is

$$xy^3 = y^3u + 3xy^2v = x^2yv^3 = x^2v^5 = 0.$$

3. PROOF OF THEOREMS

We consider in this section a fixed embedding $\mathscr{Y} \subset \mathbb{A}^n \times T$, and we assume throughout that $\mathscr{X}$ is defined in $\mathscr{Y}$ by the vanishing of

$$f = \prod_{k=1}^{s} f_k^{r_k}.$$

Recall that for each positive integer m we have defined

$$S_m f = \left(\prod_{r_k < m} f_k^{r_k} \right)^2 \cdot P^{2m-1} \left(\prod_{r_k \geq m} f_k^{r_k + m - 1} \right).$$

Let us denote by $\operatorname{Exp} Z(\mathscr{X}/T)$ the subscheme of $T(\mathscr{Y}/T)|_{\mathscr{X}}$ defined by the vanishing of $S_1 f$, and by $\operatorname{Exp} TS(\mathscr{X}/T)$ the subscheme defined by the vanishing of all $S_m f$; we call them the *explicit schemes*. For each nonnegative integer m let $\operatorname{Exp}_m(\mathscr{X}/T)$ be the scheme defined by

$$S_1 f = S_2 f = \cdots = S_m f = 0.$$

In particular $\operatorname{Exp}_0(\mathscr{X}/T)$ is $T(\mathscr{Y}/T)|_{\mathscr{X}}$ itself, $\operatorname{Exp}_1(\mathscr{X}/T)$ is $\operatorname{Exp} Z(\mathscr{X}/T)$, and for m sufficiently large $\operatorname{Exp}_m(\mathscr{X}/T)$ is $\operatorname{Exp} TS(\mathscr{X}/T)$. Each $\operatorname{Exp}_m(\mathscr{X}/T)$ is a cone over $\mathscr{X}$, and there are inclusions

$$\operatorname{Exp}_0(\mathscr{X}/T) \supset \operatorname{Exp}_1(\mathscr{X}/T) \supset \operatorname{Exp}_2(\mathscr{X}/T) \supset \cdots.$$

PROPOSITION 1. *Suppose $m \geqslant 1$. Then $\operatorname{Exp}_m(\mathscr{X}/T)$ is defined in $\operatorname{Exp}_{m-1}(\mathscr{X}/T)$ by the vanishing of a linear combination with positive coefficients of all monomials*

$$\Big(\prod_{r_k < m} f_k^{r_k}\Big)^2 \cdot \prod_{r_k \geqslant m} f_k^{r_k + m - 1} \cdot \frac{(Pf_j)^{2m-1}}{f_j^{2m-1}} \tag{a}$$

determined by f_j's for which $r_j \geqslant m$. Furthermore each individual monomial

$$\Big(\prod_{r_k < m,} f_k^{r_k}\Big)^2 \cdot \prod_{r_k \geqslant m} f_k^{r_k + m} \cdot \frac{(Pf_j)^{2m-1}}{f_j^{2m}} \tag{b}$$

vanishes on $\operatorname{Exp}_m(\mathscr{X}/T)$, and

$$\prod_{r_k \geqslant m} f_k \cdot S_m f \tag{c}$$

vanishes on $\operatorname{Exp}_{m-1}(\mathscr{X}/T)$.

Proof. We prove the three claims simultaneously by induction on m. Repeated use of the product rule for differentiation shows that $S_m f$ is a linear combination with positive coefficients of all monomials

$$\Big(\prod_{r_k < m} f_k^{r_k}\Big)^2 \cdot \prod_{r_k \geqslant m} \{ f_k^{a_{k0}} (Pf_k)^{a_{k1}} (P^2 f_k)^{a_{k2}} \cdots (P^{2m-1} f_k)^{a_{k,2m-1}} \}, \tag{1}$$

where the exponents are nonnegative integers satisfying

$$a_{k0} + a_{k1} + a_{k2} + \cdots + a_{k,2m-1} = r_k + m - 1 \tag{2}$$

for each k, and

$$\sum_{r_k \geqslant m} \{ a_{k1} + 2a_{k2} + 3a_{k3} + \cdots + (2m-1)\, a_{k,2m-1} \} = 2m - 1. \tag{3}$$

For $m=1$, the vanishing of a positive linear combination of the monomials named in (a) is clear.

For $m>1$, many of the monomials (1) are divisible by f, hence vanish on $\mathrm{Exp}_0(\mathscr{X}/T)$. Suppose we have a monomial not divisible by f, so that $a_{i0}<r_i$ for some i. Note that $r_i \geqslant m$, and that (2) and (3) together imply $r_i - a_{i0} \leqslant m$. Our monomial must be divisible by

$$\left(\prod_{r_k < r_i - a_{i0}} f_k^{r_k}\right)^2 \cdot \prod_{r_k \geqslant r_i - a_{i0}} f_k^{r_k + r_i - a_{i0}} \cdot \frac{(Pf_i)^{2(r_i - a_{i0}) - 1}}{f_i^{2(r_i - a_{i0})}}, \tag{4}$$

as we see by comparing the exponents on each f_k:

(a) if $r_k < r_i - a_{i0}$, the corresponding exponents are equal;

(b) if $r_i - a_{i0} \leqslant r_k < m$, then $r_k + r_i - a_{i0} \leqslant 2r_k$;

(c) if $r_k \geqslant m$ and $k \neq i$, then together (2) and (3) imply that $a_{k0} + a_{i0} \geqslant r_k + r_i + 2(m-1) - (2m-1)$, hence $r_k + r_i - a_{i0} \leqslant a_{k0}$;

(d) if $k = i$, the corresponding exponents are equal;

and the exponents on Pf_i:

(e) together (2) and (3) imply $2a_{i0} + a_{i1} \geqslant 2r_i - 1$, hence $2(r_i - a_{i0}) - 1 \leqslant a_{i1}$.

If $r_i - a_{i0} < m$, then (4) vanishes on $\mathrm{Exp}_{r_i - a_{i0}}(\mathscr{X}/T)$ by the inductive hypothesis, specifically the vanishing of (b); hence our monomial vanishes on $\mathrm{Exp}_m(\mathscr{X}/T)$. The opposite condition $r_i - a_{i0} = m$ forces $a_{i1} = 2m-1$ and all other terms on the left side of (3) to vanish. Hence this condition determines one of the monomials named in (a).

Repeat the argument with $S_m f$ replaced by

$$\frac{1}{f_j} \prod_{r_k \geqslant m} f_k \cdot S_m f.$$

In this case there is just one exceptional monomial, the one named in (b). Repeat the argument yet again, applying it to

$$\prod_{r_k \geqslant m} f_k \cdot S_m f.$$

In this case there are no exceptional monomials, so that the function vanishes already on $\mathrm{Exp}_{m-1}(\mathscr{X}/T)$. ∎

Using Proposition 1(a) we can easily study the localization along a component of $\mathscr{X}$. We now assume that each factor f_k is irreducible. Then over the localization along $f_k = 0$ the cone $\mathrm{Exp}_m(\mathscr{X}/T)$ is defined in

$\mathrm{Exp}_{m-1}(\mathscr{X}/T)$ by

$$f_k^{r_k-m}(Pf_k)^{2m-1}=0,$$

and in $T(\mathscr{Y}/T)|_{\mathscr{X}}$ by

$$f_k^{r_k-j}(Pf_k)^{2j-1}=0, \qquad 1\leqslant j\leqslant m. \tag{5}$$

In particular the explicit cone $\mathrm{Exp}\, TS(\mathscr{X}/T)$ is defined in $T(\mathscr{Y}/T)|_{\mathscr{X}}$ by the equations

$$f_k^{r_k-m}(Pf_k)^{2m-1}=0, \qquad 1\leqslant m\leqslant r_k. \tag{6}$$

PROPOSITION 2. *Suppose that $\mathscr{Y}\subset\mathbb{A}^n\times T$ is smooth over the product of nonsingular varieties $T=T_1\times T_2$, and that $\mathscr{X}$ is a hypersurface in $\mathscr{Y}$, flat over T. Let X_0 denote the fiber of $\mathscr{X}$ over $O\in T_2$; it is a scheme over T_1. Then the fiber over O of the relative cone* $\mathrm{Exp}\, Z(\mathscr{X}/T)$ *is the relative cone* $\mathrm{Exp}\, Z(X_O/T_1)$.

Assume furthermore that the components of $\mathscr{X}$ do not coalesce over $O\in T_2$. Then on the fiber over O the cone $\mathrm{Exp}\, Z(\mathscr{X}/T)$ *is flat over T_2; on the fiber over O the cone* $\mathrm{Exp}\, TS(\mathscr{X}/T)$ *is flat over T_2; and the fiber over O of* $\mathrm{Exp}\, TS(\mathscr{X}/T)$ *is* $\mathrm{Exp}\, TS(X_O/T_1)$.

Proof. Let f_O denote the restriction of f to Y_O, the fiber of $\mathscr{Y}$ over $O\in T_2$. Clearly $(Pf)_O=P(f_O)$. Hence the fiber over O of $\mathrm{Exp}\, Z(\mathscr{X}/T)$ is $\mathrm{Exp}\, Z(X_O/T_1)$. If furthermore the components of $\mathscr{X}$ do not coalesce over O, then $(S_m f)_O=S_m(f_O)$ for all m. Hence under this hypothesis the fiber over O of $\mathrm{Exp}\, TS(\mathscr{X}/T)$ is $\mathrm{Exp}\, TS(X_O/T_1)$.

We will prove the flatness of $\mathrm{Exp}\, Z(\mathscr{X}/T)$ and $\mathrm{Exp}\, TS(\mathscr{X}/T)$ by an induction, using the intermediate schemes $\mathrm{Exp}_m(\mathscr{X}/T)$. Since $\mathscr{X}$ is flat over T, $\mathrm{Exp}_0(\mathscr{X}/T)$ is flat over T_2.

For the inductive step, consider the surjection of structure sheaves

$$\mathcal{O}_{\mathrm{Exp}_{m-1}(\mathscr{X}/T)}\to\mathcal{O}_{\mathrm{Exp}_m(\mathscr{X}/T)}$$

coming from the inclusion of $\mathrm{Exp}_m(\mathscr{X}/T)$ into $\mathrm{Exp}_{m-1}(\mathscr{X}/T)$. We claim that the kernel is isomorphic to the quotient

$$\mathscr{Q}_{\mathscr{X}/T}=\mathcal{O}_{T(\mathscr{Y}/T)}\Big/\prod_{r_k\geqslant m} f_k,$$

i.e., that there is an exact sequence

$$0\to\mathscr{Q}_{\mathscr{X}/T}\to\mathcal{O}_{\mathrm{Exp}_{m-1}(\mathscr{X}/T)}\to\mathcal{O}_{\mathrm{Exp}_m(\mathscr{X}/T)}\to 0. \tag{7}$$

To see this, map $\mathcal{O}_{T(\mathscr{Y}/T)}$ to $\mathcal{O}_{\mathrm{Exp}_{m-1}(\mathscr{X}/T)}$ by sending 1 to $S_m f$. Suppose that

$g \cdot S_m f$ vanishes on $\mathrm{Exp}_{m-1}(\mathscr{X}/T)$. For each k with $r_k \geqslant m$ localize along $f_k = 0$. Then

$$g \cdot f_k^{r_k - m}(Pf_k)^{2m-1}$$

vanishes on the localized $\mathrm{Exp}_{m-1}(\mathscr{X}/T)$. Using (5), with m replaced by $m-1$, we see that f_k must divide g. The exactness now follows from Proposition 1(c).

To prove the flatness of $\mathrm{Exp}_m(\mathscr{X}/T)$, it suffices to prove the flatness over each curve in T_2; hence we assume that T_2 is a curve. Let t be a local parameter for T_2 at O. We can localize the exact sequence (7) over O, obtaining

$$0 \to (\mathscr{Q}_{\mathscr{X}/T})_O \to (\mathscr{O}_{\mathrm{Exp}_{m-1}(\mathscr{X}/T)})_O \to (\mathscr{O}_{\mathrm{Exp}_m(\mathscr{X}/T)})_O \to 0.$$

If the components of $\mathscr{X}$ do not coalesce over O, then we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathscr{Q}_{\mathscr{X}/T})_O \arrow[r] \arrow[d] & (\mathscr{O}_{\mathrm{Exp}_{m-1}(\mathscr{X}/T)})_O \arrow[r] \arrow[d] & (\mathscr{O}_{\mathrm{Exp}_m(\mathscr{X}/T)})_O \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathscr{Q}_{\mathscr{X}/T})_O \arrow[r] \arrow[d] & (\mathscr{O}_{\mathrm{Exp}_{m-1}(\mathscr{X}/T)})_O \arrow[r] \arrow[d] & (\mathscr{O}_{\mathrm{Exp}_m(\mathscr{X}/T)})_O \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{Q}_{X_O/T_1} \arrow[r] & \mathscr{O}_{\mathrm{Exp}_{m-1}(X_O/T_1)} \arrow[r] & \mathscr{O}_{\mathrm{Exp}_m(X_O/T_1)} \arrow[r] & 0.
\end{tikzcd}$$

Each row is the localized exact sequence just described; the vertical homomorphisms are multiplication by t and specialization. By the inductive hypothesis $\mathrm{Exp}_{m-1}(\mathscr{X}/T)$ is flat over T_2 on the fiber over O. Therefore the first two columns of the diagram are short exact. Hence in the last column multiplication by t is an injective endomorphism. That is, on the fiber over O, $\mathrm{Exp}_m(\mathscr{X}/T)$ is flat over T_2. In particular $\mathrm{Exp}\, Z(\mathscr{X}/T)$ and $\mathrm{Exp}\, TS(\mathscr{X}/T)$ are flat. ∎

PROPOSITION 3. *With notations as in Proposition 2, the fiber over $O \in T_2$ of $TS(\mathscr{X}/T)$ contains $TS(X_O/T_1)$. If the components of $\mathscr{X}$ do not coalesce over O and if $TS(\mathscr{X}/T)$ is Cohen–Macaulay, then the fiber over O of $TS(\mathscr{X}/T)$ is $TS(X_O/T_1)$.*

Proof. Let $\mathscr{I}$ be the ideal sheaf of the relative diagonal in $\mathscr{X} \times_T \mathscr{X}$. The natural morphism of sheaves

$$\bigoplus_{j \geqslant 0} \frac{\mathscr{I}^j}{\mathscr{I}^{j+1}} \otimes \mathscr{O}_{X_O} \to \bigoplus_{j \geqslant 0} \frac{(\mathscr{I} \otimes \mathscr{O}_{X_O})^j}{(\mathscr{I} \otimes \mathscr{O}_{X_O})^{j+1}}$$

is surjective. Hence $TS(X_O/T_1)$ is a closed subscheme of $TS(\mathscr{X}/T)$.

By the device described in Section 1, we may assume that $TS(\mathscr{X}/T)$ has no irrelevant components, hence that we may recover it from the blowup $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$ of $\mathscr{X} \times_T \mathscr{X}$ along $\mathscr{X}$. The projectivized tangent star cone $\mathbb{P}TS(\mathscr{X}/T)$ is the exceptional divisor of the blowup; if $\mathbb{P}TS(\mathscr{X}/T)$ is Cohen–Macaulay then so is the blowup. The fiber over $O \in T_2$ of $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$ can differ from $Bl_{X_O}(X_O \times_{T_1} X_O)$ only over the diagonal X_O. Any extra component must lie over X_O and inside the exceptional divisor of $Bl_{Y_O}(Y_O \times_{T_1} Y_O)$, where Y_O denotes the fiber over O of $\mathscr{Y}$; the dimension of such a component is at most twice the dimension of X_O. Hence the codimension in $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$ of the fiber over O equals the number of defining equations; if the blowup is Cohen–Macaulay then so is the fiber over O. Furthermore any component over the diagonal X_O must map onto an entire component.

Now suppose that the components of $\mathscr{X}$ do not coalesce over O. As usual suppose that $\mathscr{X}$ is defined by the vanishing of

$$f = \prod_{k=1}^{s} f_k^{r_k};$$

let

$$f_{\text{red}} = \prod_{k=1}^{s} f_k.$$

Consider a generic (hence nonsingular) point x on a component of X_O, and a generic tangent line at x in the ambient space Y_O. The restriction of f_{red} to this line vanishes just once. If, however, this line were in the fiber of $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$ over x, i.e., if this line were a limiting secant line, two zeros of f_{red} would be coalescing at x, so that f_{red} would vanish more than once. Hence this line is not contained in the fiber of $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$ over x; hence the fiber of $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$ over $O \in T_2$ has no component over X_O. Since this fiber is Cohen–Macaulay it has no embedded components. Therefore this fiber is precisely $Bl_{X_O}(X_O \times_{T_1} X_O)$, and the fiber over O of $TS(\mathscr{X}/T)$ is $TS(X_O/T_1)$. ∎

PROPOSITION 4. *Suppose that $\mathscr{Y} \subset \mathbb{A}^n \times T$ is smooth over the nonsingular variety T, and that $\mathscr{X}$ is a hypersurface in $\mathscr{Y}$, flat over T. Then $TS(\mathscr{X}/T)$ is a closed subscheme of* Exp $TS(\mathscr{X}/T)$.

Proof. Again we may assume that $TS(\mathscr{X}/T)$ has no irrelevant components, hence that we may recover it from $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$. This blowup is contained in the blowup of $\mathbb{A}^n \times \mathbb{A}^n \times T$ along its relative diagonal $\mathbb{A}^n \times T$. We will use $x_1, ..., x_n$ as coordinates on the first factor of $\mathbb{A}^n \times \mathbb{A}^n \times T$ and $\bar{x}_1, ..., \bar{x}_n$ as coordinates on the second factor. We use $u_1, ..., u_n$ as blowup coordinates, so that the (larger) blowup is defined inside $\mathbb{A}^n \times \mathbb{A}^n \times T \times \mathbb{P}^{n-1}$ by the equations

$$u_i(\bar{x}_j - x_j) = u_j(\bar{x}_i - x_i).$$

Without loss of generality we restrict our attention to the chart $u_1 = 1$, so that $\bar{x}_j - x_j = u_j(\bar{x}_1 - x_1)$.

Consider the scheme $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X}) \times \mathbb{A}^1$, using a coordinate λ on the second factor. Let

$$g = \prod_{r_k < m} f_k^{r_k}$$

and

$$h = \prod_{r_k \geq m} f_k^{r_k + m - 1},$$

so that

$$f = g \cdot \prod_{r_k \geq m} f_k^{r_k}$$

and

$$S_m f = g^2 \cdot P^{2m-1} h.$$

By a (finite) Taylor series expansion we have

$$h(x_1 + \lambda u_1, ..., x_n + \lambda u_n) = \sum_{d=0} \frac{\lambda^d}{d!} P^d h(x_1, ..., x_n). \tag{8}$$

By the product rule for differentiation the first m terms on the right are divisible by

$$\prod_{r_k \geq m} f_k^{r_k}.$$

Hence on $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X}) \times \mathbb{A}^1$ the function

$$g(x_1, ..., x_n) \cdot g(\bar{x}_1, ..., \bar{x}_n) \cdot h(x_1 + \lambda u_1, ..., x_n + \lambda u_n)$$

is divisible by λ^m. This function is the same as

$$g(x_1, ..., x_n) \cdot g(\bar{x}_1, ..., \bar{x}_n) \cdot h(\bar{x}_1 + (\lambda - \bar{x}_1 + x_1) u_1, ..., \bar{x}_n + (\lambda - \bar{x}_1 + x_1) u_n),$$

which by the same argument is divisible by $(\lambda - \bar{x}_1 + x_1)^m$. On

$\mathbb{P}TS(\mathscr{X}/T)\times\mathbb{A}^1$, where $\bar{x}_1=x_1$, this function must therefore be divisible by λ^{2m}. Again using (8), we see that

$$S_m f = g^2 \cdot P^{2m-1}h$$

must vanish on $\mathbb{P}TS(\mathscr{X}/T)\times\mathbb{A}^1$. It therefore vanishes on the projectivized tangent star $\mathbb{P}TS(\mathscr{X}/T)$.

(This argument was suggested by the discussion in [5, p. 10 ff.].) ∎

PROPOSITION 5. *Under the same hypotheses,* Exp $TS(\mathscr{X}/T)$ *is Cohen–Macaulay.*

Proof. The equations in (6) describe Exp $TS(\mathscr{X}/T)$ over the localization along an irreducible component of $\mathscr{X}$, and imply that each component of Exp $TS(\mathscr{X}/T)$ has codimension at least 2 in $T(\mathscr{Y}/T)$. To compute the depth of Exp $TS(\mathscr{X}/T)$ we use the exact sequence

$$0\to\mathscr{Q}_{\mathscr{X}/T}\to\mathscr{O}_{\mathrm{Exp}_{m-1}(\mathscr{X}/T)}\to\mathscr{O}_{\mathrm{Exp}_m(\mathscr{X}/T)}\to 0 \tag{7}$$

obtained in the proof of Proposition 2. Auslander and Buchsbaum's formula [1, 2.3] (cf. [4, Theorem 19.1]) tells us that

$$\operatorname{depth}\mathscr{M}=\operatorname{depth}\mathscr{O}_{T(\mathscr{Y}/T)}\text{-projective dimension }\mathscr{M}$$

for each module $\mathscr{M}$ over the localization at a point p of $\mathscr{O}_{T(\mathscr{Y}/T)}$. The localizations at p of the hypersurface structure sheaves $\mathscr{Q}_{\mathscr{X}/T}$ and $\mathscr{O}_{\mathrm{Exp}_0(\mathscr{X}/T)}$ have projective dimension $\leqslant 1$. Inductively we prove that the localization at p of each $\mathscr{O}_{\mathrm{Exp}_m(\mathscr{X}/T)}$ has projective dimension $\leqslant 2$, by considering the long exact sequence for Ext and (7). In particular $\mathscr{O}_{\mathrm{Exp}\,TS(\mathscr{X}/T)}$ has projective dimension $\leqslant 2$. Hence Exp $TS(\mathscr{X}/T)$ is Cohen–Macaulay. ∎

PROPOSITION 6. *Suppose furthermore that $\mathscr{X}$ is a divisor with normal crossings (not necessarily reduced). Then $TS(\mathscr{X}/T)=$* Exp *$TS(\mathscr{X}/T)$.*

Proof. By Proposition 4, $TS(\mathscr{X}/T)$ is a closed subscheme of Exp $TS(\mathscr{X}/T)$. By Proposition 5, Exp $TS(\mathscr{X}/T)$ has no embedded components. Hence to show that $TS(\mathscr{X}/T)$ and Exp $TS(\mathscr{X}/T)$ are the same it suffices to show that they have the same multiplicity along each irreducible component of Exp $TS(\mathscr{X}/T)$.

Since Exp $TS(\mathscr{X}/T)$ has codimension 1 in $T(\mathscr{Y}/T)|_{\mathscr{X}}$, each irreducible component of Exp $TS(\mathscr{X}/T)$ maps onto either an irreducible component of $\mathscr{X}$ or an intersection of two such components. Recall that over the localization along a component $f_k=0$ of $\mathscr{X}$, Exp $TS(\mathscr{X}/T)$ is defined in $T(\mathscr{Y}/T)$ by $f_k^{r_k}=0$ and

$$f_k^{r_k-m}(Pf_k)^{2m-1}=0,\qquad 1\leqslant m\leqslant r_k. \tag{6}$$

Hence over this component of $\mathscr{X}$ there is one component of Exp $TS(\mathscr{X}/T)$, with multiplicity equal to the length of the local ring generated by f_k and Pf_k, and subject to the preceding relations; this length is r_k^2.

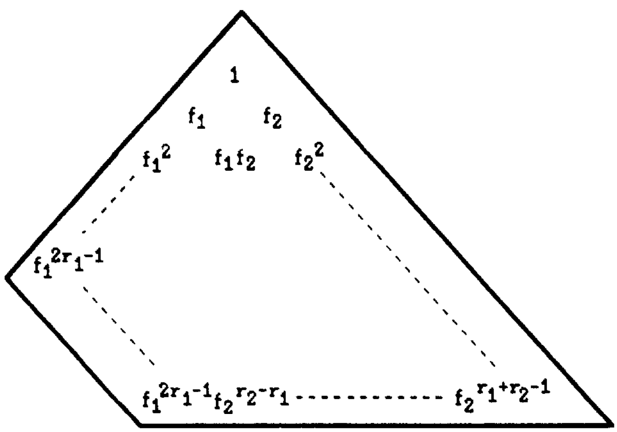
Similarly we localize over a normal crossing $f_1 = f_2 = 0$. Note that in the localization Pf_1 and Pf_2 are units. By Proposition 1, Exp $TS(\mathscr{X}/T)$ is defined in $T(\mathscr{Y}/T)$ by the vanishing of

(a) $f_1^{r_1} f_2^{r_2}$;

(b) for each m with $1 \leqslant m \leqslant r_1$, a linear combination with positive coefficients of $f_1^{r_1-m} f_2^{r_2+m-1}(Pf_1)^{2m-1}$ and $f_1^{r_1+m-1} f_2^{r_2-m}(Pf_2)^{2m-1}$; and

(c) $f_1^{2r_1} f_2^{r_2-m}$, $r_1 < m \leqslant r_2$.

In particular $f_1^{2r_1}$ vanishes. In (b) we have r_1 independent linear combinations of the $2r_1$ monomials $f_1^{2r_1-1} f_2^{r_2-r_1}$, $f_1^{2r_1-2} f_2^{r_2-r_1+1}$, ..., $f_2^{r_1+r_2-1}$. We conclude that Exp $TS(\mathscr{X}/T)$ has one component over the normal crossing; its multiplicity is obtained by subtracting r_1 from the number of monomials in the following trapezoid:



(If $r_1 = r_2$ we have a triangle of monomials.) The multiplicity is $2r_1 r_2$.

In calculating the multiplicities of the components of $TS(\mathscr{X}/T)$ we may use the device described in Section 1, viz., replacing $\mathscr{X}$ if necessary by $\mathscr{X} \times \mathbb{A}^1$. This device guarantees that $\mathscr{X}$ is nowhere dense in $\mathscr{X} \times_T \mathscr{X}$. Hence the components of the blowup of $\mathscr{X} \times_T \mathscr{X}$ along the diagonal are in one-to-one correspondence with those of $\mathscr{X} \times_T \mathscr{X}$ itself. The fundamental class of the blowup of $\mathscr{X} \times_T \mathscr{X}$ along its diagonal is

$$[Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})] = \sum_{k,l} r_k r_l [Bl_{\mathscr{X}_k \cap \mathscr{X}_l}(\mathscr{X}_k \times_T \mathscr{X}_l)],$$

where $\mathscr{X}_k$ denotes the component $f_k=0$. For the exceptional divisor $\mathbb{P}TS(\mathscr{X}/T)$ we have

$$[\mathbb{P}TS(\mathscr{X}/T)]=\sum_{k,l} r_k r_l [\mathbb{P}C_{\mathscr{X}_k \cap \mathscr{X}_l}(\mathscr{X}_k \times_T \mathscr{X}_l)],$$

where the schemes on the right are projectivized normal cones. In fact for our divisor $\mathscr{X}$ with normal crossings each of these schemes is the projectivization of a normal bundle to a nonsingular subvariety, hence a variety. ∎

Proof of Theorem 3. Given a family $\mathscr{X}_1 \subset \mathscr{Y}_1$ of hypersurfaces over T_1 defined by

$$0=f(x_1, x_2, ..., x_n, t)=\prod_{k-1}^{s} f_k^{r_k},$$

we enlarge it to a family $\mathscr{X} \subset \mathscr{Y}_1 \times \mathbb{A}^1$ over $T=T_1 \times \mathbb{A}^1$ as follows. We let $g_1, ..., g_s$ be regular functions of $x_1, ..., x_n, t$ defining a collection of divisors with normal crossings in $\mathscr{Y}_1$; we then define $\mathscr{X}$ by

$$0=F(x_1, x_2, ..., x_n, t, \alpha)=\prod_{k=1}^{s} (f_k+\alpha g_k)^{r_k},$$

where α is a coordinate on the affine line. Then $\mathscr{X}$ is a relative hypersurface over T, whose fiber over a generic point of $\mathbb{A}^1$ is a normal crossings divisor. One immediately verifies that the components of $\mathscr{X}$ do not coalesce over $\alpha=0$. Noncoalescence is an open condition, so the components do not coalesce over a generic point of $\mathbb{A}^1$.

By Proposition 4, $TS(\mathscr{X}/T)$ is contained in Exp $TS(\mathscr{X}/T)$. For a generic point t of $\mathbb{A}^1$ we have, by Propositions 2, 6, and 3,

$$\begin{aligned}\text{fiber of Exp } TS(\mathscr{X}/T) &= \text{Exp } TS(X_t/T_1)\\ &= TS(X_t/T_1)\\ &\subset \text{fiber of } TS(\mathscr{X}/T),\end{aligned}$$

so that the two fibers are equal. Since Exp $TS(\mathscr{X}/T)$ is flat over $\mathbb{A}^1$, $TS(\mathscr{X}/T)=\text{Exp } TS(\mathscr{X}/T)$.

In particular the two schemes have identical fibers over O. By Proposition 5, $TS(\mathscr{X}/T)$ is Cohen–Macaulay; hence by Proposition 3 its fiber over O is $TS(\mathscr{X}_1/T_1)$. By Proposition 2 the fiber of Exp $TS(\mathscr{X}/T)$ is Exp $TS(\mathscr{X}_1/T_1)$. We conclude that $TS(\mathscr{X}_1/T_1)=\text{Exp } TS(\mathscr{X}_1/T_1)$.

The equality of these cones implies equality of each summand in the grading, in particular the degree 1 summand. Hence we have $Z(\mathscr{X}_1/T_1)=\text{Exp } Z(\mathscr{X}_1/T_1)$. ∎

Proof of Theorem 1. This is immediate from Theorem 3 and from Proposition 2, taking $T_1 = \operatorname{Spec} k$. ∎.

Proof of Theorem 2. As usual we assume $TS(\mathscr{X}/T)$ has no irrelevant components. Suppose the components of $\mathscr{X}$ coalesce along a component of X_O. We reverse the argument in the proof of Proposition 3. Consider a generic point x on a component of X_O, and a generic tangent line at x in the ambient space Y_O. The restriction of f_{red} to this line vanishes more than once. Hence this tangent line is contained in the fiber over O of $Bl_{\mathscr{X}}(\mathscr{X} \times_T \mathscr{X})$; hence this blowup has a component over the diagonal X_O. The fiber over O of the exceptional divisor $\mathbb{P}TS(X/T)$ is therefore too large; $\mathbb{P}TS(X/T)$ cannot be flat and its fiber cannot be $\mathbb{P}TS(X_O)$. ∎

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